

Information-geometrical characterization of statistical models which are statistically equivalent to probability simplexes

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Abstract—The probability simplex is the set of all probability distributions on a finite set and is the most fundamental object in the finite probability theory. In this paper we give a characterization of statistical models on finite sets which are statistically equivalent to probability simplexes in terms of α -families including exponential families and mixture families. The subject has a close relation to some fundamental aspects of information geometry such as α -connections and autoparallelity.

I. AN INTRODUCTORY EXAMPLE

Let $\mathcal{X} = \{0, 1, 2\}$ and let $M = \{p_\lambda \mid 0 < \lambda < 1\}$ be the set of probability distributions on \mathcal{X} of the form

$$p_\lambda = (p_\lambda(0), p_\lambda(1), p_\lambda(2)) = (\lambda, (1 - \lambda)/2, (1 - \lambda)/2).$$

The statistical model M has the following three properties. Firstly, it is a mixture family since

$$p_\lambda = \lambda(1, 0, 0) + (1 - \lambda)(0, 1/2, 1/2).$$

Secondly, it is an exponential family since

$$\log p_\lambda = \theta F - \psi(\theta),$$

where $\theta = \log(2\lambda/(1 - \lambda))$, $(F(0), F(1), F(2)) = (1, 0, 0)$ and $\psi(\theta) = -\log(1 - \lambda)/2 = \log(2 + e^\theta)$. Lastly, M is statistically equivalent to the 1-dimensional open probability simplex $\mathcal{P}_1 = \{(\lambda, 1 - \lambda) \mid 0 < \lambda < 1\}$ in the sense that there exist a channel V from $\{0, 1\}$ to \mathcal{X} and a channel W from \mathcal{X} to $\{0, 1\}$ such that M is the set of output distributions of V for input distributions in \mathcal{P}_1 and that V is invertible by W . The matrix representations of these channels are given by

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Note that the invertibility $WV = I$ holds.

Our aim is to show the equivalence between the first two properties and the last one.

II. STATEMENT OF THE MAIN RESULT

We begin with giving some basic definitions which are necessary to state our problem.

For an arbitrary finite set \mathcal{X} , let $\overline{\mathcal{P}}(\mathcal{X})$ and $\mathcal{P}(\mathcal{X})$ be the sets of probability distributions and of strictly positive probability distributions on \mathcal{X} ;

$$\overline{\mathcal{P}}(\mathcal{X}) := \{p \mid p : \mathcal{X} \rightarrow [0, 1], \sum_x p(x) = 1\}$$

$$\mathcal{P}(\mathcal{X}) := \{p \mid p : \mathcal{X} \rightarrow (0, 1), \sum_x p(x) = 1\}.$$

In particular, let for an arbitrary positive integer d

$$\overline{\mathcal{P}}_d := \overline{\mathcal{P}}(\{0, 1, \dots, d\})$$

$$\mathcal{P}_d := \mathcal{P}(\{0, 1, \dots, d\}),$$

which we call the d -dimensional (closed and open) probability simplexes.

A mapping $\Gamma : \overline{\mathcal{P}}(\mathcal{X}) \rightarrow \overline{\mathcal{P}}(\mathcal{Y})$, where \mathcal{X} and \mathcal{Y} are finite sets, is called a *Markov map* when there exists a channel $W(y|x)$ from \mathcal{X} to \mathcal{Y} such that, for any $p \in \overline{\mathcal{P}}(\mathcal{X})$,

$$\Gamma(p) = \sum_x W(\cdot | x)p(x).$$

i.e., $\Gamma(p)$ is the output distribution of the channel W corresponding to the input distribution p . Note that a Markov map is affine; $\Gamma(\lambda p + (1 - \lambda)q) = \lambda\Gamma(p) + (1 - \lambda)\Gamma(q)$ for $\forall p, q \in \overline{\mathcal{P}}(\mathcal{X})$ and $0 \leq \lambda \leq 1$.

Let M and N be smooth submanifolds (statistical models) of $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$, respectively. When there exist a pair of Markov maps $\Gamma : \overline{\mathcal{P}}(\mathcal{X}) \rightarrow \overline{\mathcal{P}}(\mathcal{Y})$ and $\Delta : \overline{\mathcal{P}}(\mathcal{Y}) \rightarrow \overline{\mathcal{P}}(\mathcal{X})$ such that their restrictions $\Gamma|_M$ and $\Delta|_N$ are bijections between M and N and are the inverse mappings of each other, we say that M and N are *Markov equivalent* or *statistically equivalent* and write as $M \simeq N$.

The aim of this paper is to give a characterization of statistical models which are statistically equivalent to probability simplexes. The main result is as follows.

Theorem 1 For an arbitrary smooth submanifold M of $\mathcal{P}(\mathcal{X})$, the following conditions are mutually equivalent.

- (i) $M \simeq \mathcal{P}_d$, where $d = \dim M$.
- (ii) M is an exponential family and is a mixture family.
- (iii) $\exists \alpha \neq \exists \beta$, M is an α -family and is an β -family.
- (iv) $\forall \alpha$, M is an α -family.

Explanation of exponential family, mixture family and α -family for arbitrary $\alpha \in \mathbb{R}$ as well as the proof of the theorem will be presented in subsequent sections. Here we only give a few remarks on condition (i). Firstly, (i) is equivalent to the condition that $\exists d'$, $M \simeq \mathcal{P}_{d'}$, since if $M \simeq \mathcal{P}_{d'}$ then M and $\mathcal{P}_{d'}$ must be diffeomorphic, so that $\dim M = \dim \mathcal{P}_{d'} = d'$. Secondly, (i) is equivalent to the condition $\overline{M} \simeq \overline{\mathcal{P}_d}$, where \overline{M} denotes the topological closure of M , and means that \overline{M} is the set of output distributions of an invertible (erro-free) channel.

III. SOME FACTS ABOUT CONDITION (i)

From the definition of the relation \simeq , condition (i) implies that there exist $\Gamma : \overline{\mathcal{P}(\mathcal{X})} \rightarrow \overline{\mathcal{P}_d}$ and $\Delta : \overline{\mathcal{P}_d} \rightarrow \overline{\mathcal{P}(\mathcal{X})}$ satisfying $\Gamma \circ \Delta = \text{id}$ (the identity on $\overline{\mathcal{P}_d}$). Let $\{q_0, q_1, \dots, q_d\} \subset \overline{\mathcal{P}(\mathcal{X})}$ be defined by

$$\Delta(\delta_i) = q_i, \quad \forall i \in \{0, 1, \dots, d\}, \quad (1)$$

where δ_i is the delta distributions on $\{0, 1, \dots, d\}$ concentrated on i . Then it is easy to see, as is shown in Lemma 9.5 and its ‘‘Supplement’’ of [1] where our Δ is called a *congruent embedding* (of $\overline{\mathcal{P}_d}$ into $\overline{\mathcal{P}(\mathcal{X})}$), that the supports $A_i := \text{supp}(q_i)$ constitute a partition of \mathcal{X} in the sense that

$$A_i \cap A_j = \emptyset \text{ if } i \neq j, \text{ and } \bigcup_{i=0}^d A_i = \mathcal{X}, \quad (2)$$

and the left inverse Γ of Δ is represented as

$$\Gamma(p) = \sum_{i=0}^d p(A_i) \delta_i, \quad \forall p \in \overline{\mathcal{P}(\mathcal{X})}, \quad (3)$$

where $p(A_i) := \sum_{x \in A_i} p(x)$. In addition, condition (i) implies $M = \Delta(\mathcal{P}_d) := \{\Delta(\lambda) \mid \lambda \in \mathcal{P}_d\}$, so that from (1) we have

$$M = \left\{ \sum_{i=0}^d \lambda_i q_i \mid (\lambda_0, \dots, \lambda_d) \in \mathcal{P}_d \right\}. \quad (4)$$

Conversely, if a statistical model $M \subset \mathcal{P}(\mathcal{X})$ is represented in the form (4) by a collection of $d+1$ distributions $\{q_i\}$ on \mathcal{X} whose supports $\{A_i\}$ constitute a partition of \mathcal{X} , then we see that M satisfies condition (i) by defining Δ and Γ by (1) and (3). Thus a necessary and sufficient condition for (i) is obtained, which will be used in later arguments to prove the theorem.

IV. α -FAMILY, e-FAMILY AND m-FAMILY

Following the way developed in [5] (see also [3], [4]), we give the definition of α -family, which includes that of *exponential family* and *mixture family* as special cases.

For an arbitrary $\alpha \in \mathbb{R}$, define a function $L^{(\alpha)} : \mathbb{R}^+ (= (0, \infty)) \rightarrow \mathbb{R}$ by¹

$$L^{(\alpha)}(u) = \begin{cases} u^{\frac{1-\alpha}{2}} & (\alpha \neq 1) \\ \log u & (\alpha = 1). \end{cases} \quad (5)$$

The function $L^{(\alpha)}$ is naturally extended to a mapping $(\mathbb{R}^+)^{\mathcal{X}} \rightarrow \mathbb{R}^{\mathcal{X}}$ ($f \mapsto L^{(\alpha)}(f)$) by

$$\left(L^{(\alpha)}(f) \right)(x) = L^{(\alpha)}(f(x)). \quad (6)$$

For a submanifold M of $\mathcal{P}(\mathcal{X})$, its *denormalization* \tilde{M} is defined by

$$\tilde{M} := \{ \tau p \mid p \in M \text{ and } \tau \in \mathbb{R}^+ \}, \quad (7)$$

where τp denotes the function $\mathcal{X} \ni x \mapsto \tau p(x) \in \mathbb{R}^+$. The denormalization is an extended manifold obtained by relaxing the normalization constraint $\sum_x p(x) = 1$. Obviously, \tilde{M} is a submanifold of $\widetilde{\mathcal{P}(\mathcal{X})}$, and $\mathcal{P}(\mathcal{X}) = (\mathbb{R}^+)^{\mathcal{X}}$ is an open subset of $\mathbb{R}^{\mathcal{X}}$. When the image

$$L^{(\alpha)}(\tilde{M}) = \left\{ L^{(\alpha)}(\tau p) \mid p \in M \text{ and } \tau \in \mathbb{R}^+ \right\}$$

forms an open subset of an affine subspace, say Z , of $\mathbb{R}^{\mathcal{X}}$, M is called an α -family. In this paper, it is assumed for simplicity that M is maximal in the sense that

$$L^{(\alpha)}(\tilde{M}) = Z \cap L^{(\alpha)}((\mathbb{R}^+)^{\mathcal{X}}). \quad (8)$$

Since it follows from the definition (5) of $L^{(\alpha)}$ that

$$L^{(\alpha)}((\mathbb{R}^+)^{\mathcal{X}}) = \begin{cases} (\mathbb{R}^+)^{\mathcal{X}} & (\alpha \neq 1) \\ \mathbb{R}^{\mathcal{X}} & (\alpha = 1), \end{cases}$$

(8) is written as

$$L^{(\alpha)}(\tilde{M}) = \begin{cases} Z \cap (\mathbb{R}^+)^{\mathcal{X}} & (\alpha \neq 1) \\ Z & (\alpha = 1). \end{cases} \quad (9)$$

Note that, as is pointed out in section 2.6 of [4], an affine subspace Z satisfying (9) must be a linear subspace when $\alpha \neq 1$. Note also that $\mathcal{P}(\mathcal{X})$ is an α -family for $\forall \alpha \in \mathbb{R}$, corresponding to the case when $Z = \mathbb{R}^{\mathcal{X}}$.

When $\alpha = 1$, the notion of α -family is equivalent to that of exponential family, whose general form is $M = \{p_\theta \mid \theta = (\theta^1, \dots, \theta^d) \in \mathbb{R}^d\}$ such that

$$p_\theta(x) = \exp \left[C(x) + \sum_{i=1}^d \theta^i F_i(x) - \psi(\theta) \right], \quad (10)$$

where C, F_1, \dots, F_d are functions on \mathcal{X} and ψ is a function on \mathbb{R}^d defined by

$$\psi(\theta) = \log \sum_x \exp \left[C(x) + \sum_{i=1}^d \theta^i F_i(x) \right]. \quad (11)$$

¹ $L^{(\alpha)}(u)$ can be replaced with $aL^{(\alpha)}(u) + b$ by arbitrary constants $a \neq 0$ and b , possibly depending on α . In [3], [4], [5], these constants are properly chosen so that the $\pm\alpha$ -duality and the limit of $\alpha \rightarrow 1$ can be treated in a convenient way.

When $\alpha = -1$, on the other hand, the notion of α -family is equivalent to that of mixture family, whose general form is $M = \{p_\theta \mid \theta = (\theta^1, \dots, \theta^d) \in \Theta\}$ such that

$$p_\theta(x) = C(x) + \sum_{i=1}^d \theta^i F_i(x), \quad (12)$$

where F_1, \dots, F_d are functions on \mathcal{X} satisfying $\sum_x F_i(x) = 0$ and $\Theta := \{\theta \in \mathbb{R}^d \mid \forall x, p_\theta(x) > 0\}$.

When $\alpha \neq 1$, the general form of α -family $M = \{p_\theta \mid \theta = (\theta^1, \dots, \theta^d) \in \Theta\}$ is

$$p_\theta(x) = \left\{ \sum_{j=0}^d \xi^j(\theta) F_j(x) \right\}^{\frac{2}{1-\alpha}}. \quad (13)$$

See §2.6 of [4] for further details.

V. PROOF OF (i) \Rightarrow (iv)

Assume (i), which implies that there exists a collection of $d+1$ probability distributions $\{q_i\} \subset \overline{\mathcal{P}}(\mathcal{X})$ whose supports $\{A_i\}$ constitute a partition of \mathcal{X} and that M is represented as (4). Then the denormalization \tilde{M} is represented as

$$\tilde{M} = \left\{ \sum_{i=0}^d \lambda_i q_i \mid (\lambda_0, \dots, \lambda_d) \in (\mathbb{R}^+)^{d+1} \right\}. \quad (14)$$

Let α be an arbitrary real number such that $\alpha \neq 1$. Since $L^{(\alpha)}(0) = 0$ in this case, it follows from the disjointness of the supports of $\{q_i\}$ that

$$L^{(\alpha)} \left(\sum_i \lambda_i q_i \right) = \sum_i \lambda_i^{\frac{1-\alpha}{2}} L^{(\alpha)}(q_i)$$

for any $(\lambda_0, \dots, \lambda_d) \in (\mathbb{R}^+)^{d+1}$. From this we have

$$\begin{aligned} L^{(\alpha)}(\tilde{M}) &= \left\{ \sum_{i=0}^d \lambda_i^{\frac{1-\alpha}{2}} L^{(\alpha)}(q_i) \mid (\lambda_0, \dots, \lambda_d) \in (\mathbb{R}^+)^{d+1} \right\} \\ &= \left\{ \sum_{i=0}^d \xi_i L^{(\alpha)}(q_i) \mid (\xi_0, \dots, \xi_d) \in (\mathbb{R}^+)^{d+1} \right\} \\ &= Z \cap (\mathbb{R}^+)^{\mathcal{X}}, \end{aligned}$$

where Z is the $(d+1)$ -dimensional linear subspace of $\mathbb{R}^{\mathcal{X}}$ spanned by $L^{(\alpha)}(q_i)$, $i \in \{0, 1, \dots, d\}$. This proves that M is an α -family for any $\alpha \neq 1$.

Let $\alpha = 1$. For any $x \in \mathcal{X}$, we have

$$\begin{aligned} L^{(1)} \left(\sum_i \lambda_i q_i \right) (x) &= \log \left(\sum_i \lambda_i q_i(x) \right) \\ &= \log(\lambda_j q_j(x)) \\ &= \log \lambda_j + \log q_j(x) \\ &= \sum_i (\log \lambda_i + \log q_i(x)) 1_{A_i}(x), \end{aligned}$$

where j denotes the element of $\{0, 1, \dots, d\}$ such that $x \in A_j$. Letting $C \in \mathbb{R}^{\mathcal{X}}$ be defined by $C(x) = \sum_i (\log q_i(x)) 1_{A_i}(x)$, we have

$$\begin{aligned} L^{(1)}(\tilde{M}) &= \left\{ C + \sum_{i=0}^d (\log \lambda_i) 1_{A_i} \mid (\lambda_0, \dots, \lambda_d) \in (\mathbb{R}^+)^{d+1} \right\} \\ &= \left\{ C + \sum_{i=0}^d \xi_i 1_{A_i} \mid (\xi_0, \dots, \xi_d) \in \mathbb{R}^{d+1} \right\}, \end{aligned}$$

which is an affine subspace of $\mathbb{R}^{\mathcal{X}}$. This proves that M is a 1-family (an exponential family).

The implication (i) \Rightarrow (iv) has thus been proved.

VI. EQUIVALENCE OF (ii), (iii) AND (iv)

The implications (iv) \Rightarrow (ii) \Rightarrow (iii) are obvious. To see (iii) \Rightarrow (iv), some results of information geometry are invoked.

Remark 1: The notion of affine connections appears only in this section. Since the implication (ii) \Rightarrow (i) will be proved in the next section without using affine connections (at least explicitly), we do not need them in proving the equivalence of the conditions of Theorem 1 except for (iii).

We first introduce some concepts from general differential geometry. Let S be a smooth manifold and denote by $\mathcal{T}(S)$ the set of smooth vector fields on S . Here, by a vector field on S we mean a mapping, say X , such that $X : S \ni p \mapsto X_p \in T_p(S)$, where $T_p(S)$ denotes the tangent space of S at p . An affine connection on S is represented by a mapping $\nabla : \mathcal{T}(S) \times \mathcal{T}(S) \ni (X, Y) \mapsto \nabla_X Y \in \mathcal{T}(S)$, which is called a covariant derivative, satisfying certain conditions. Let M be a smooth submanifold of S . Then ∇ is naturally defined on $\mathcal{T}(M) \times \mathcal{T}(M)$, so that $\nabla_X Y$ is defined for any vector fields on M . However, the value $\nabla_X Y$ in this case is a mapping $M \ni p \mapsto (\nabla_X Y)_p \in T_p(S)$ in general and is not a vector field on M (i.e., $\nabla_X Y \notin \mathcal{T}(M)$) unless

$$(\nabla_X Y)_p \in T_p(M), \quad \forall p \in M. \quad (15)$$

When (15) holds for $\forall X, Y \in \mathcal{T}(M)$, M is said to be *autoparallel* w.r.t. ∇ or ∇ -*autoparallel* in S .

Let ∇, ∇' and ∇'' be affine connection on S for which there exists a real number a satisfying²

$$\nabla'' = a\nabla + (1-a)\nabla'. \quad (16)$$

If a submanifold M is ∇ -autoparallel and ∇' -autoparallel, then it is also ∇'' -autoparallel. This implication is obvious from $(\nabla''_X Y)_p = a(\nabla_X Y)_p + (1-a)(\nabla'_X Y)_p$ and the autoparallelity condition (15), which will be invoked later.

As was independently introduced by Čencov [1] and Amari [2], a one-parameter family of affine connections, which are called the α -connections ($\alpha \in \mathbb{R}$), are defined on a manifold

²For arbitrary affine connections ∇ and ∇' , their affine combination $a\nabla + (1-a)\nabla'$ always becomes an affine connection.

of probability distributions. After Amari's notation, the α -connection is written in the form of affine combination

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla^{(1)} + \frac{1-\alpha}{2}\nabla^{(-1)}, \quad (17)$$

which implies that

$$\nabla^{(\gamma)} = \frac{\gamma-\beta}{\alpha-\beta}\nabla^{(\alpha)} + \frac{\alpha-\gamma}{\alpha-\beta}\nabla^{(\beta)} \quad (18)$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha \neq \beta$.

When a submanifold M of S is autoparallel w.r.t. the α -connection in S , we say that M is α -autoparallel in S . Since (18) is of the form (16), it follows that if M is α -autoparallel and β -autoparallel in S for some $\alpha \neq \beta$, then it is γ -autoparallel in S for all $\gamma \in \mathbb{R}$. On the other hand, it was shown in [5] (see also section 2.6 of [4]) that, for any submanifold M of $\mathcal{P}(\mathcal{X})$ and for any real number α , M is an α -family if and only if M is α -autoparallel in $\mathcal{P}(\mathcal{X})$. Combination of these two results proves (iii) \Rightarrow (iv).

Remark 2: Since the e-connection and the m-connection are dual w.r.t. the Fisher information metric [3], [4], [5], condition (ii) is a special case of doubly autoparallelity introduced by Ohara; see [6], [7] and the reference cited there. It is pointed out in [7] that the α -autoparallelity for all α follows from that for $\alpha = \pm 1$.

VII. PROOF OF (ii) \Rightarrow (i)

Assume (ii), which means that there exist two affine subspaces $Z^{(e)}$ and $Z^{(m)}$ of $\mathbb{R}^{\mathcal{X}}$ such that

$$L^{(e)}(\tilde{M}) = \{\log \mu \mid \mu \in \tilde{M}\} = Z^{(e)} \quad (19)$$

$$L^{(m)}(\tilde{M}) = \tilde{M} = Z^{(m)} \cap (\mathbb{R}^+)^{\mathcal{X}}, \quad (20)$$

where $L^{(e)} := L^{(1)}$ and $L^{(m)} := L^{(-1)}$. Let $V^{(e)}$ and $V^{(m)}$ be the linear spaces of translation vectors of $Z^{(e)}$ and $Z^{(m)}$, respectively, so that we have $Z^{(e)} = f + V^{(e)}$ for any $f \in Z^{(e)}$ and $Z^{(m)} = g + V^{(m)}$ for any $g \in Z^{(m)}$ ³.

Lemma 1 $V^{(e)}$ is closed w.r.t. multiplication of functions; i.e., $a, b \in V^{(e)} \Rightarrow ab \in V^{(e)}$, where the product ab is defined by $(ab)(x) = a(x)b(x)$.

Proof. The map

$$\Phi := L^{(e)}|_{\tilde{M}} : \tilde{M} \ni \mu \mapsto \log \mu \in Z^{(e)}$$

is a diffeomorphism from $\tilde{M} = Z^{(m)} \cap (\mathbb{R}^+)^{\mathcal{X}}$, which is an open subset of $Z^{(m)}$, onto $Z^{(e)}$. The differential map of Φ at a point $\mu \in \tilde{M}$ is defined by

$$(\mathrm{d}\Phi)_{\mu} \left(\frac{d\mu(t)}{dt} \Big|_{t=0} \right) = \frac{d}{dt} \Phi(\mu(t)) \Big|_{t=0}$$

for any smooth curve $\mu(t)$ in \tilde{M} and is represented as

$$(\mathrm{d}\Phi)_{\mu} : V^{(m)} \ni f \mapsto \frac{f}{\mu} \in V^{(e)}.$$

³Actually, $Z^{(m)}$ is a linear space as mentioned in section IV, and therefore $Z^{(m)} = V^{(m)}$.

This gives a linear isomorphism from $V^{(m)}$ onto $V^{(e)}$. Therefore, for any two points $\mu, \nu \in \tilde{M}$, we can define

$$(\mathrm{d}\Phi)_{\nu} \circ (\mathrm{d}\Phi)_{\mu}^{-1} : V^{(e)} \ni a \mapsto \frac{\mu a}{\nu} \in V^{(e)}.$$

This means that, for any $a \in V^{(e)}$ and any $\mu, \nu \in \tilde{M}$, we have $\frac{\mu a}{\nu} \in V^{(e)}$. For arbitrary $a \in V^{(e)}$ and $\nu \in \tilde{M}$, let us define a map $\Psi_{a,\nu}$ by

$$\Psi_{a,\nu} : \tilde{M} \ni \mu \mapsto \frac{\mu a}{\nu} \in V^{(e)}.$$

Then its differential at a point $\mu \in \tilde{M}$ is given by

$$(\mathrm{d}\Psi_{a,\nu})_{\mu} : V^{(m)} \ni g \mapsto \frac{ga}{\nu} \in V^{(e)}.$$

Composing this map with the inverse of

$$(\mathrm{d}\Phi)_{\nu} : V^{(m)} \ni g \mapsto \frac{g}{\nu} \in V^{(e)},$$

we have

$$(\mathrm{d}\Psi_{a,\nu})_{\mu} \circ (\mathrm{d}\Phi)_{\nu}^{-1} : V^{(e)} \ni b \mapsto ab \in V^{(e)}.$$

This proves that $a, b \in V^{(e)} \Rightarrow ab \in V^{(e)}$. \square

Lemma 2 $V^{(e)}$ contains the constant functions on \mathcal{X} .

Proof. From the definition (7) of \tilde{M} , for any $\mu \in \tilde{M}$ and any positive constant $\tau = e^c$, we have $\tau\mu \in \tilde{M}$. This implies that both $\log \mu$ and $\log(\tau\mu)$ belong to $Z^{(e)}$, and hence the translation $\log(\tau\mu) - \log \mu = \log \tau = c$ belongs to $V^{(e)}$. \square

These two lemmas state that $V^{(e)}$ is a subalgebra of the commutative algebra $\mathbb{R}^{\mathcal{X}}$ with the unit element 1 (: the constant function $x \mapsto 1$) of $\mathbb{R}^{\mathcal{X}}$ contained in $V^{(e)}$. From a well known result on such subalgebras⁴, it is concluded that there exists a partition $\{A_i\}_{i=0}^d$ of \mathcal{X} such that

$$V^{(e)} = \left\{ \sum_{i=0}^d c_i 1_{A_i} \mid (c_0, \dots, c_d) \in \mathbb{R}^{d+1} \right\}. \quad (21)$$

Let an element p_0 of $M (\subset \tilde{M})$ be arbitrarily fixed. Then we have

$$Z^{(e)} = \log p_0 + V^{(e)}. \quad (22)$$

From (19), (21) and (22) and the disjointness of $\{A_i\}$, we have

$$\begin{aligned} \tilde{M} &= \{\mu \mid \log \mu \in Z^{(e)}\} \\ &= \{\mu \mid \log \mu - \log p_0 \in V^{(e)}\} \\ &= \left\{ \mu \mid \exists (c_0, \dots, c_d) \in \mathbb{R}^{d+1}, \right. \\ &\quad \left. \log \mu = \log p_0 + \sum_{i=0}^d c_i 1_{A_i} \right\}, \\ &= \left\{ p_0 \sum_{i=0}^d e^{c_i} 1_{A_i} \mid (c_0, \dots, c_d) \in \mathbb{R}^{d+1} \right\} \\ &= \left\{ \sum_{i=0}^d \lambda_i q_i \mid (\lambda_0, \dots, \lambda_d) \in (\mathbb{R}^+)^{d+1} \right\}, \end{aligned}$$

⁴Although various mathematical extensions of this result including infinite-dimensional and/or noncommutative versions are known, the author of the present paper could find no appropriate reference describing the result for the finite-dimensional commutative case with an elementary proof. So, we give a proof in the appendix for the readers' sake.

where

$$q_i := \frac{1}{p_0(A_i)} p_0 1_{A_i}, \quad i \in \{0, \dots, d\}.$$

Then $\{q_i\}$ are probability distributions on \mathcal{X} whose supports are $\text{supp}(q_i) = A_i$, and

$$\begin{aligned} M &= \tilde{M} \cap \mathcal{P}(\mathcal{X}) \\ &= \left\{ \sum_{i=0}^d \lambda_i q_i \mid (\lambda_0, \dots, \lambda_d) \in \mathcal{P}_d \right\}. \end{aligned}$$

Since this is the same form as (4), condition (i) has been derived.

VIII. CONCLUSION

We have shown Theorem 1 which gives an information-geometrical characterization of statistical models on finite sample spaces which are statistically equivalent to open probability simplexes \mathcal{P}_d . The statistical equivalence (also called the Markov equivalence) to probability simplexes played a crucial role in Čencov's pioneering work [1] on information geometry, where the notions of Fisher information metric and the α -connections were characterized in terms of the statistical equivalence. The present work shed another light on the relation between the statistical equivalence and information geometry.

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REFERENCES

- [1] N. N. Čencov (Chentsov), *Statistical Decision Rules and Optimal Inference*, AMS, 1982 (original Russian edition: Nauka, Moscow, 1972).
- [2] S. Amari, "Differential geometry of curved exponential families—curvature and information loss", *The Annals of Statistics*, 10, 357–385, 1982.
- [3] S. Amari, *Differential-Geometrical Methods in Statistics*, Springer, Lecture Notes in Statistics 28, 1985.
- [4] S. Amari and H. Nagaoka, *Methods of information geometry*, AMS & OUP, 2000.
- [5] H. Nagaoka and S. Amari, "Differential geometry of smooth families of probability distributions", Technical Report METR 82-7, Dept. of Math. Eng. and Instr. Phys, Univ. of Tokyo, 1982. (<http://www.keisu.t.u-tokyo.ac.jp/research/techrep/data/1982/METR82-07.pdf>)
- [6] A. Ohara, "Information geometric analysis of an interior point method for semidefinite programming", *Geometry in Present Day Science* (eds. O. E. Barndorff-Nielsen and E. B. V. Jensen), pp.49-74, World Scientific, 1999.
- [7] A. Ohara, "Geodesics for dual connections and means on symmetric cones", *Integr. equ. oper. theory*, 50, 537–548, 2004.

APPENDIX

Proposition Let \mathcal{X} be a finite set and V be a subalgebra of $\mathbb{R}^{\mathcal{X}}$ containing the constant functions. Then there exists a partition $\{A_i\}_{i=1}^n$ of \mathcal{X} such that

$$V = \left\{ \sum_{i=1}^n c_i 1_{A_i} \mid (c_1, \dots, c_n) \in \mathbb{R}^n \right\}. \quad (23)$$

Proof. Let

$$\mathcal{B} := \{f^{-1}(\lambda) \mid \lambda \in \mathbb{R}, f \in V\} \subset 2^{\mathcal{X}}, \quad (24)$$

which is the totality of the level sets $f^{-1}(\lambda) = \{x \mid f(x) = \lambda\} \subset \mathcal{X}$ of functions in V . We first show that, for any $B \in \mathcal{B}$,

$$B \in \mathcal{B} \Leftrightarrow 1_B \in V. \quad (25)$$

Since \Leftarrow is obvious, it suffices to show \Rightarrow . Assume $B \in \mathcal{B}$, so that $B = f^{-1}(\lambda)$ for some $f \in V$ and $\lambda \in \mathbb{R}$. When B is the empty set ϕ , we have $1_B = 0 \in V$. So we assume $B \neq \phi$, which means that $\lambda \in f(\mathcal{X})$. Let the elements of $f(\mathcal{X})$ be $\lambda_0, \lambda_1, \dots, \lambda_k$, where $\lambda_0 = \lambda$ and $\lambda_i \neq \lambda_j$ if $i \neq j$, and let $B_i := f^{-1}(\lambda_i)$. Then we have $f = \sum_{i=0}^k \lambda_j 1_{B_i}$ with $B_0 = B$. Let $a(t) = a_0 t^k + a_1 t^{k-1} + \dots + a_k$ be a polynomial satisfying $a(\lambda_0) = 1$ and $a(\lambda_i) = 0$ for any $i \neq 0$. Explicitly, $a(t)$ is expressed as

$$a(t) = \prod_{i=1}^k \frac{t - \lambda_i}{\lambda_0 - \lambda_i}.$$

It follows that

$$a(f) = \sum_{i=0}^k a(\lambda_i) 1_{B_i} = 1_{B_0} = 1_B.$$

In addition, $a(f) = a_0 f^k + a_1 f^{k-1} + \dots + a_k$ belongs to V since V is a subalgebra of $\mathbb{R}^{\mathcal{X}}$ with $1 \in V$. Hence we have $1_B \in V$.

Using (25), we see that

$$\mathcal{X} \in \mathcal{B}, \quad (26)$$

$$B \in \mathcal{B} \Rightarrow B^c \in \mathcal{B}, \quad (27)$$

$$B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2 \in \mathcal{B} \quad (28)$$

as

$$1_{\mathcal{X}} = 1 \in V \Rightarrow \mathcal{X} \in \mathcal{B}, \quad (29)$$

$$\begin{aligned} B \in \mathcal{B} \Rightarrow 1_B \in V \Rightarrow 1_{B^c} &= 1 - 1_B \in V \\ &\Rightarrow B^c \in \mathcal{B}, \end{aligned} \quad (30)$$

$$\begin{aligned} B_1, B_2 \in \mathcal{B} \Rightarrow 1_{B_1}, 1_{B_2} \in V \Rightarrow 1_{B_1 \cap B_2} &= 1_{B_1} 1_{B_2} \in V \\ &\Rightarrow B_1 \cap B_2 \in \mathcal{B}. \end{aligned} \quad (31)$$

Properties (26)-(28) implies that \mathcal{B} is an additive class of sets (σ -algebra) on the finite entire set \mathcal{X} . Therefore, \mathcal{B} is generated by a partition $\{A_1, \dots, A_n\}$ of \mathcal{X} in the sense that every element of \mathcal{B} is the union of some (or no) elements of $\{A_1, \dots, A_n\}$. Recalling the definition (24) of \mathcal{B} , we conclude (23). \square